Global differential equations for slice-regular functions

R. Ghiloni, A. Perotti

Department of Mathematics
University of Trento, Italy

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Outline

Alternative *-algebra

A = real alternative *-algebra with unity of finite dimension d, i.e. equipped with a linear anti–involution $x \mapsto x^c$:

$$\begin{cases} (x^c)^c = x & \forall x \in A, \\ (xy)^c = y^c x^c & \forall x, y \in A, \\ x^c = x & \text{for every real } x. \end{cases}$$

$t(x) := x + x^c \in A$ is the trace of x, $\quad n(x) := xx^c \in A$ is the norm of x

The quadratic cone of A

$$\mathcal{Q}_A := \mathbb{R} \cup \{x \in A \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R},\ 4n(x) > t(x)^2\}$$

- $\mathcal{Q}_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$ and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_A$, $I \neq \pm J$
- $\mathcal{Q}_A = A \Leftrightarrow A \simeq \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$

Assume $\mathbb{S}_A := \{J \in \mathcal{Q}_A \mid J^2 = -1\} \neq \emptyset$ (square roots of -1)

Stem and slice functions

$A \otimes \mathbb{C} = \{w = x + iy \mid x, y \in A\}$, with $i^2 = -1$ and complex conjugation $\overline{w} = \overline{x + iy} = x - iy$

Definition

Let $D \subseteq \mathbb{C}$, invariant under complex conjugation. If $F : D \longrightarrow A \otimes \mathbb{C}$ satisfies $F(\overline{z}) = \overline{F(z)}$ for every $z \in D$, then F is called a stem function on D

circular sets

$$\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha, \beta \in \mathbb{R},\ \alpha + i\beta \in D,\ J \in \mathbb{S}_A\} \subseteq \mathcal{Q}_A$$

Stem and slice functions

Definition

Any stem function $F = F_1 + iF_2 : D \longrightarrow A \otimes \mathbb{C}$ induces a (left) slice function $f = \mathcal{I}(F) : \Omega_D \longrightarrow A$. If $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$

$$f(x) := F_1(z) + JF_2(z) \qquad (z = \alpha + i\beta).$$

i.e. $\Phi_J : A \otimes \mathbb{C} \to A, \Phi_J(a + ib) := a + Jb$
makes the diagram commute $\forall J \in \mathbb{S}_A$

$$\begin{array}{ccc} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \supseteq D & \xrightarrow{F} & A \otimes_{\mathbb{R}} \mathbb{C} \\ \downarrow \Phi_J & & \downarrow \Phi_J \\ \Omega_D & \xrightarrow{f} & A \end{array}$$

Slice derivatives. Slice-regular functions

$\mathcal{S}^0(\Omega_D, A)$ the $\mathbb{R}$-vector space of slice functions induced by continuous functions, $\mathcal{S}^1(\Omega_D, A) = \{f = \mathcal{I}(F) \in \mathcal{S}^0(\Omega_D, A) \mid F \in \mathscr{C}^1(D, A \otimes \mathbb{C})\}$
We can define slice derivative operators:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial x^c} : \mathcal{S}^1(\Omega_D, A) \longrightarrow \mathcal{S}^0(\Omega_D, A)$$

$$\frac{\partial f}{\partial x} := \mathcal{I}\left(\frac{1}{2}\left(\frac{\partial F}{\partial \alpha} - i\frac{\partial F}{\partial \beta}\right)\right), \quad \frac{\partial f}{\partial x^c} := \mathcal{I}\left(\frac{1}{2}\left(\frac{\partial F}{\partial \alpha} + i\frac{\partial F}{\partial \beta}\right)\right)$$

Definition

A slice function $f \in \mathcal{S}^1(\Omega_D, A)$ is slice regular if

$$\frac{\partial f}{\partial x^c} = 0 \text{ on } \Omega_D.$$

$\mathcal{SR}(\Omega_D, A)$ the $\mathbb{R}$-vector space of slice regular functions on Ω_D

Slice derivatives. Slice-regular functions

Remarks

(1) Every slice-regular function is real analytic, and it holds:

$$\frac{\partial}{\partial x}\big(\mathcal{SR}(\Omega_D, A)\big) \subset \mathcal{SR}(\Omega_D, A).$$

(2) A slice function f in $\mathcal{S}^1(\Omega_D, A)$ is slice regular if and only if

$$\frac{\partial f}{\partial x^c} = 0 \quad \text{on } \Omega_D \setminus \mathbb{R}$$

(3) $\dfrac{\partial}{\partial x}$ extends the notion of Cullen derivative ∂_C (Gentili-Struppa) to any A and to any slice (not necessarily regular) function

Slice derivatives. Slice-regular functions

The definition of ∂_C can be given also on $\Omega_D \subseteq A$. Moreover, on $\Omega_D \setminus \mathbb{R}$, the definition can be extended to any real differentiable function:
let $y \in \Omega_D \setminus \mathbb{R}$. Write $y = \xi + \eta J$, where $\xi, \eta \in \mathbb{R}$ with $\eta > 0$ and $J \in \mathbb{S}_A$.
Define $w := \xi + i\eta \in D$, $\Phi_J(\alpha + i\beta) := \alpha + \beta J$ and

$$\partial_C f(y) := \frac{1}{2}\left(\frac{\partial}{\partial\alpha} - J\frac{\partial}{\partial\beta}\right)(f \circ \Phi_J)(w)$$

$$\overline{\partial}_C f(y) := \frac{1}{2}\left(\frac{\partial}{\partial\alpha} + J\frac{\partial}{\partial\beta}\right)(f \circ \Phi_J)(w)$$

Remark

On $\Omega_D \setminus \mathbb{R}$, the derivatives $\partial f/\partial x$ and $\partial_C f$ of a slice function $f \in \mathcal{S}^1(\Omega_D, A)$ coincide. The same for $\partial f/\partial x^c$ and $\overline{\partial}_C f$

The global differential operators ϑ and $\overline{\vartheta}$

We construct two global differential operators

$$\vartheta, \overline{\vartheta} : \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A) \longrightarrow \mathscr{C}^0(\Omega_D \setminus \mathbb{R}, A)$$

that extend the restrictions to $\Omega_D \setminus \mathbb{R}$ of $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial x^c}$ (that are defined only on slice functions).

Example (Quaternionic space $\mathbb{H}$)

For $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$: the operator $\overline{\vartheta}$ has the form

$$\overline{\vartheta} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\operatorname{Im}(x)}{n(\operatorname{Im}(x))} \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}\right)$$

Cf. Colombo, González-Cervantes, Sabadini for the related operator

$$G = n(\operatorname{Im}(x))\,\overline{\vartheta}$$

The global differential operators ϑ and $\overline{\vartheta}$

Let

$$\mathcal{S}^1_*(\Omega_D, A) := \{g \in \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A) \mid g = f_{|\,\Omega_D \setminus \mathbb{R}} \text{ for some } f \in \mathcal{S}^1(\Omega_D, A)\}$$

Then $\mathcal{S}^1_*(\Omega_D, A) \simeq \mathcal{S}^1(\Omega_D, A)$. The same for $\mathcal{S}^0_*(\Omega_D, A) \simeq \mathcal{S}^0(\Omega_D, A)$.

$$\begin{array}{ccc} \mathcal{S}^1(\Omega_D, A) & \xrightarrow{\partial/\partial x^c} & \mathcal{S}^0(\Omega_D, A) \\ r_1 \downarrow \uparrow ext_1 & & r_0 \downarrow \uparrow ext_0 \\ \mathcal{S}^1_*(\Omega_D, A) & \xrightarrow{\partial/\partial x^c_*} & \mathcal{S}^0_*(\Omega_D, A) \end{array}$$

The global differential operators ϑ and $\overline{\vartheta}$

Remark

f is slice regular $\iff r_1 \circ f \in \ker(\partial/\partial x_*^c)$ i.e.

$$\mathcal{SR}(\Omega_D, A) = ext_1\big(\ker(\partial/\partial x_*^c)\big)$$

$$\mathcal{SR}_*(\Omega_D, A) := r_1\big(\mathcal{SR}(\Omega_D, A)\big) = \ker(\partial/\partial x_*^c)$$

We now construct, explicitly in coordinates, $\vartheta,\ \overline{\vartheta} : \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A)$ to $\mathscr{C}^0(\Omega_D \setminus \mathbb{R}, A)$ which extend $\partial/\partial x_*$ and $\partial/\partial x_*^c$, respectively.
Fix a basis $\mathcal{B} = (u_0, u_1, \ldots, u_{d-1})$ of A such that $u_0 = 1$ and

$$t(x) = 0 \ \text{ for every } x \in \langle u_1, \ldots, u_{d-1}\rangle \cap \mathcal{Q}_A.$$

Lemma

*Every alternative *-algebra A has a basis $\mathcal{B} = (u_0, u_1, \ldots, u_{d-1})$ satisfying these properties*

The global differential operators ϑ and $\overline{\vartheta}$

$A \simeq_{\mathcal{B}} \mathbb{R}^d$ with product $x \cdot y$: for every $\mathrm{x} = (\mathrm{x}_0, \mathrm{x}_1, \ldots, \mathrm{x}_{d-1}) \in \Omega_D$, let

$$\mathrm{Re}(\mathrm{x}) := \frac{t(\mathrm{x})}{2} = \frac{\mathrm{x} + \mathrm{x}^c}{2} \quad \text{and} \quad \mathrm{Im}(\mathrm{x}) := \frac{\mathrm{x} - \mathrm{x}^c}{2}$$

Then $\mathrm{Re}(\mathrm{x}) = (\mathrm{x}_0, 0, \ldots, 0)$ and $\mathrm{Im}(\mathrm{x}) = (0, \mathrm{x}_1, \ldots, \mathrm{x}_{d-1})$.
Define $\vartheta, \overline{\vartheta} : \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A) \longrightarrow \mathscr{C}^0(\Omega_D \setminus \mathbb{R}, A)$ by setting

$$\vartheta := \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{x}_0} - \frac{\mathrm{Im}(\mathrm{x})}{n(\mathrm{Im}(\mathrm{x}))} \cdot \left(\sum_{\ell=1}^{d-1} \mathrm{x}_\ell \frac{\partial}{\partial \mathrm{x}_\ell}\right)\right)$$

and

$$\overline{\vartheta} := \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{x}_0} + \frac{\mathrm{Im}(\mathrm{x})}{n(\mathrm{Im}(\mathrm{x}))} \cdot \left(\sum_{\ell=1}^{d-1} \mathrm{x}_\ell \frac{\partial}{\partial \mathrm{x}_\ell}\right)\right)$$

The global differential operators ϑ and $\overline{\vartheta}$

Theorem

The differential operators $\vartheta, \overline{\vartheta}$ are well-defined, and it holds:

(i) *If $f \in \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A)$, then $\vartheta f = \partial_C f$ and $\overline{\vartheta} f = \overline{\partial}_C f$.*

(ii) *If $f \in \mathcal{S}^1_*(\Omega_D, A)$, then $\dfrac{\partial f}{\partial x_*} = \vartheta f$ and $\dfrac{\partial f}{\partial x_*^c} = \overline{\vartheta} f$.*

In particular, we have that

$$\mathcal{SR}_*(\Omega_D, A) = \ker(\overline{\vartheta}) \cap \mathcal{S}^1_*(\Omega_D, A)$$

$$\mathcal{SR}(\Omega_D, A) = \mathrm{ext}_1\big(\ker(\overline{\vartheta})\big) \cap \mathcal{S}^1(\Omega_D, A)$$

Remark

The equality $\mathcal{SR}_*(\Omega_D, A) = \ker(\overline{\vartheta})$ holds only in the special case in which $D \cap \mathbb{R} = \emptyset$ and $|\mathbb{S}_A| = 2$

The kernel of $\overline{\vartheta}$: slice (regular) or not slice (regular)?

Theorem

- *If* $D \cap \mathbb{R} = \emptyset$ *and* $|\mathbb{S}_A| = 2$*, then* $\ker(\overline{\vartheta}) \subset \mathcal{S}^1_*(\Omega_D, A)$ *and hence*

$$\mathcal{SR}_*(\Omega_D, A) = \ker(\overline{\vartheta})$$

- *If* $D \cap \mathbb{R} \neq \emptyset$ *or* $|\mathbb{S}_A| > 2$*, then* $\ker(\overline{\vartheta}) \not\subset \mathcal{S}^1_*(\Omega_D, A)$ *and hence*

$$\mathcal{SR}_*(\Omega_D, A) \subsetneq \ker(\overline{\vartheta})$$

- *Suppose D is connected. Then, given* $f \in \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A)$*, the following assertions are equivalent:*
  - (*i*) $f \in \mathcal{SR}_*(\Omega_D, A)$*, i.e. f extends to a slice regular function on* Ω_D
  - (*ii*) *f belongs to* $\ker(\overline{\vartheta})$ *and it admits a continuous extension on the whole* Ω_D

Application: slice-regular homogeneous polynomials

Assume that the real components of $f : \mathbb{H} \longrightarrow \mathbb{H}$ are homogeneous polynomials of degree m in the real variables x_0, x_1, x_2, x_3. Then

$$2\,\overline{\vartheta} f = \partial_{x_0} f + \frac{x - x_0}{n(x - x_0)}(mf - x_0 \partial_{x_0} f) = \frac{x - x_0}{n(x - x_0)}(mf - x\partial_{x_0} f).$$

Therefore, $\overline{\vartheta} f = 0$ if and only if $mf = x\,\partial_{x_0} f$. Inductively we get

$$\overline{\vartheta} f = 0 \quad \Rightarrow \quad m!\, f(x) = x^m \frac{\partial^m f}{\partial x_0^m}.$$

This means that $f(x)$ is of the form

$$f(x) = x^m a, \quad \text{with } a \in \mathbb{H} \text{ constant.}$$

Using the Theorem, we can say that the only slice-regular homogeneous polynomials on $\mathbb{H}$ are those of the form $x^m a$ ($a \in \mathbb{H}$ constant)

Examples: quaternions, Clifford algebras, octonions

Clifford algebra $\mathbb{R}_n$ (idem for $\mathbb{R}_{p,q}$): the standard basis $\mathcal{B} = \{e_K\}_{K\in\mathcal{P}(n)}$ given by the products $e_K := e_{k_1}\cdots e_{k_p}$ satisfies the required conditions

$$\overline{\vartheta} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\operatorname{Im}(x)}{n(\operatorname{Im}(x))}\sum_{|K|\equiv 1,2 \pmod 4} x_K \frac{\partial}{\partial x_K}\right).$$

The restriction of $\overline{\vartheta}$ to the space of paravectors

$$\mathbb{R}^{n+1} = \{x = x_0 + \textstyle\sum_{i=1}^n x_i e_i \in \mathbb{R}_n \,|\, x_0, x_1, \ldots, x_n \in \mathbb{R}\} \subset \mathcal{Q}_n$$

has already been considered by Colombo, González-Cervantes, Sabadini for the operator (related to slice monogenic functions)

$$G = n(\operatorname{Im}(x))\,\overline{\vartheta}_{|\mathbb{R}^{n+1}}$$

See also the book of Gürlebeck, Habetha, Sprössig, under the name of radial differential operators

Examples: quaternions, Clifford algebras, octonions

$\mathbb{R}_2 \simeq \mathbb{H}$. After the identification $e_1 = i$, $e_2 = j$, $e_{12} = ij = k$, in the coordinates (x_0, x_1, x_2, x_3) of a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$, then

$$\overline{\vartheta} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\mathrm{Im}(x)}{n(\mathrm{Im}(x))}\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}\right).$$

Let $\mathbb{O}$ be the non–associative division algebra of octonions. Let k be a fixed imaginary unit of $\mathbb{O}$, $\{1, i, j, ij\}$ a real basis of $\mathbb{H}$. Then $\mathcal{B} = \{1, i, j, ij, k, ik, jk, (ij)k\}$ is a basis of $\mathbb{O}$, satisfying the required conditions. If $x_0, x_1, \dots, x_7$ denote the coordinates w.r.t. $\mathcal{B}$, we can write

$$\overline{\vartheta} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\mathrm{Im}(x)}{n(\mathrm{Im}(x))}\sum_{i=1}^{7} x_i \frac{\partial}{\partial x_i}\right),$$

where $\mathrm{Im}(x) = x - x_0 = x_1 i + x_2 j + x_3 ij + x_4 k + x_5 ik + x_6 jk + x_7 (ij)k$.
See also Gentili-Struppa for octonionic Cullen derivatives

Characterization of ker($\overline{\vartheta}$)

Definition

We call the functions in $\mathscr{C}^1(\mathbb{S}_A \times D^+, A \otimes \mathbb{C})$ semi–stem functions on $D^+ = \{\alpha + i\beta \in D \,|\, \beta > 0\}$. Given a semi–stem function $\mathcal{F} = \mathcal{F}_1 + i\mathcal{F}_2$, we define

$$\mathcal{I}(\mathcal{F})(x) := \mathcal{F}_1(J,z) + J\mathcal{F}_2(J,z) : \Omega_D \setminus \mathbb{R} \longrightarrow A$$

for $x = \alpha + \beta J \in \Omega_D \setminus \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $z = \alpha + i\beta \in D^+$.
$\mathcal{F}$ is separately z–holomorphic if, for each $J \in \mathbb{S}_A$, it holds:

$$\left(\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right)\mathcal{F}(J,\cdot) = 0 \text{ on } D^+.$$

The (holomorphic) stem functions on $D \setminus \mathbb{R}$ can be identified with the (separately z–holomorphic) semi–stem functions on D^+ independent from J

Characterization of ker($\overline{\vartheta}$)

$\mathcal{I}$ maps $\mathscr{C}^1(\mathbb{S}_A \times D^+, A \otimes \mathbb{C})$ to $\mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A)$
and has a right inverse $\mathcal{I}^{-1}$ (dependent on a splitting basis of A)

Theorem

$$\ker(\overline{\vartheta}) = \{f \in \mathscr{C}^1(\Omega_D \setminus \mathbb{R}, A) \,|\, \mathcal{I}^{-1}(f) \textit{ is separately } z\textit{–holomorphic}\}$$

Remark

$\mathcal{SR}(\Omega_D \setminus \mathbb{R}, A)$ consists of all functions f, where $\mathcal{F} = \mathcal{I}^{-1}(f)$ is a separately z–holomorphic function on D^+, independent from $J \in \mathbb{S}_A$

Characterization of sliceness

Lemma

Let $f \in \mathscr{C}^0(\Omega_D, A)$, let $J \in \mathbb{S}_A$.The following assertions are equivalent:

(*i*) *f is slice*

(*ii*) *The following formula holds:*

$$f(x) = \frac{1}{2}(f(y) + f(y^c)) - \frac{I}{2}(J(f(y) - f(y^c)))$$

for each $y = \alpha + \beta J \in \Omega_D \cap \mathbb{C}_J$ and for each $x = \alpha + \beta I$, where $\alpha, \beta \in \mathbb{R}$ and $I \in \mathbb{S}_A$

Examples

Examples

- The slice-regular function on $\mathcal{Q}_A \setminus \mathbb{R}$ defined by

$$f(x) := I \quad \text{for } x = \alpha + \beta I$$

is induced by the stem function $\mathcal{I}^{-1}(f) = \mathcal{F}(z) = 0 + i \cdot 1$ but also by the semi-stem function (not J-independent) $\mathcal{F}'(J, z) = J + i \cdot 0$.

- Let $\phi : \mathbb{S}_A \to \mathbb{R}$ be nonconstant, of class C^1, such that $\phi(-J) = \phi(J)\ \forall J \in \mathbb{S}_A$. Define

$$f(x) := \phi(I) \quad \text{for } x = \alpha + \beta I \in \mathcal{Q}_A \setminus \mathbb{R}$$

Then $f \in \ker(\overline{\vartheta})$ but $f \notin \mathcal{SR}(\mathcal{Q}_A \setminus \mathbb{R}, A)$

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